Math 206B Lecture 11 Notes

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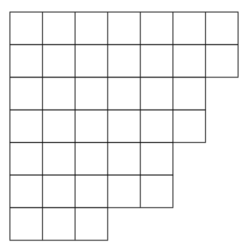
February 1, 2019

1 RSK: The Final Chapter

1.1 Diagonals in Young diagrams and generalizing geometric RSK

Let ν be a partition of k, and let $f : \nu \to \mathbb{R}_+$ be a function on squares of the Young diagram. Define the diagonal sums $\alpha_c := \sum_{i-j=c} f(i,j)$, where $c \in \mathbb{Z}$. Similarly, define $\beta_c := \sum_{i=1}^{r_c} \sum_{j=1}^{s_c} f(i,j)$, where $c \in \mathbb{Z}$.

Example 1.1. Consider the following partition.



Then

$$\alpha_{-1} = f(1,2) + f(2,3) + f(3,4) + f(4,5)$$

and $(r_c, s_c) = (5,4)$ is the position of the lowermost square on this diagonal.

Theorem 1.1. *Fix $\overline{d}$, and let*

$$P_\nu(\overline{d}) = \{f : \nu \to \mathbb{R}_+ \mid f(i,j) \leq f(i+1,j) \leq f(i,j+1), \alpha_c(f) = dc \, \forall c\},$$

$$Q_\nu(\overline{d}) = \{g : \nu \to \mathbb{R}_+ \mid \beta_c(f) = d_c \, \forall c\}.$$

Then there exists some $\Phi : P_\nu(\overline{d}) \to Q_\nu(\overline{d})$ such that Φ is

1. *piecewise linear,*

2. *volume-preserving,*

3. *continuous,*

4. $\Phi : P_\nu \cap \mathbb{Z}^L \to Q_\nu \cap \mathbb{Z}^K$.

Moreover, Φ *commutes with transposition.*

Corollary 1.1. *The number of integer points in* $P_\nu(\overline{d})$ *is the same as the number of integer points in* $Q_\nu(\overline{d})$.

Corollary 1.2 (reduction to RSK)**.**

$$\#M(\overline{a}, \overline{b}) = \sum_{\lambda} \#\,\mathrm{SSYT}(\lambda, \overline{a}) \times \#\,\mathrm{SSYT}(\lambda, \overline{b})$$

Example 1.2. Let $\nu(\ell^\ell)$ be an $\ell \times \ell$ square. If we split it up along the diagonal, we get two tableaux, a $\mathrm{SSYT}(\lambda, (d_0 - d_{-1}, d_{-1} - d_{-2}, \dots))$ and a $\mathrm{SSYT}(\lambda, (d_0 - d_1, d_1 - d_2, \dots))$. So in the case of a square, we get RSK.

1.2 Description and proof of generalized geometric RSK

Let's prove the theorem.

Proof. Proceed by induction. If $\lambda = \varnothing$, we are done, and if λ is a square, we are also done because P and Q are the same. Let $r - s = c$, so $(r_c, s_c) = (r, s)$. And let $\overline{\nu} = \nu - (r, s)$; we are removing a box from the diagram at position (r, s) on the boundary of the diagram. We have $\Phi_{\overline{\nu}} : P_{-nu} \to Q_{\overline{\nu}}$, and we want to get Φ_ν.

Draw a diagonal from the square (r, s) up and left. We want to alter boxes on the diagonal. Take ξ sending $f(i, j) \mapsto \max\{f(i-1, j), f(i, j-1)\} + \min\{f(i, +1, j), f(i, j+1)\} - f(i, j)$. Call this $\overline{f}(i, j)$. Then we get $f \mapsto \overline{f} \in P_{\overline{\nu}} \mapsto \overline{g} \in Q_{\overline{\nu}}$. How do we get $g \in Q_\nu$ from $\overline{g}$? Just add the last square by setting $g(r, s) := f(r, s) - \max\{f(r-1, s), f(r, s-1)\}$.

Why is Φ is well-defined? Note that no two adjacent diagonals can contain corners. Now think about the order of the squares we chop off and replace. If we write this order in reverse, we get a Young tableau. We claim that if Γ is a graph on $\mathrm{SYT}(\nu)$ with $(i, i+1)$ swaps allowed in distinct diagonals, then Γ_ν is connected. If we keep switching to put every number in our tableau in lexicographic order, we will eventually get the full lexicographic ordering. So the graph Γ_ν is connected, which makes this process well-defined. $\square$

Example 1.3. Take the element of P_ν

1	1	4
2	3	4
4	4	5

Chop off the 5 in the bottom right hand corner, and alter the diagonal of that 5. After replacing the space of the 5, we get

0	1	4
2	3	4
4	4	1

Now chop off the 4 on the right in the bottom row and alter its diagonal. After replacing that 4, we get

0	1	4
1	3	4
4	0	1

Continue like this, replacing one square at a time in the corner (of the diagram, only counting squares we haven't chopped off and replaced) and altering its diagonal until we've altered everything.

0	2	4
1	3	0
4	0	1

1	2	4
1	1	0
4	0	1

1	2	2
1	1	0
4	0	1

Continuing like this, we eventually get

1	1	2
0	1	0
3	0	1